The area of a circle is $\pi r^2$

We can estimate the area by inscribing triangles in the circle and summing the areas of the triangles. See the figure below with 8 inscribed triangles. Obviously the more triangles we use, the more accurate the estimate of the area.

In the circle, each triangle has a central angle $\theta = \frac{2\pi}{8}$. In general, if we use $n$ triangles, the angle will be $\frac{2\pi}{n}$.

Let's find the area of one such triangle in a circle of radius $R$.

First note that the height (h) between the angle $\theta$ and the radius $R$ is:

$$\cos(\theta) = \frac{h}{R}$$

or rearranged:

$$h = R \cos(\theta)$$

Hence, the area of the triangle is:

$$\frac{1}{2}bh = \frac{1}{2}R \sin(\frac{\pi}{4}) \cdot R \cos(\theta)$$
Consider the following circle with radius $r$ and $\theta = \frac{2\pi}{n}$.

Let's show that the area of the above inscribed triangle is

$$\frac{1}{2} r^2 \sin \left( \frac{2\pi}{n} \right).$$

First note that the height ($h$) bisects the angle $\theta$ and the base ($b$). From the definition of $\sin$ we have

$$\sin \left( \frac{\theta}{2} \right) = \frac{b}{r} \quad \text{or} \quad 2r \sin \left( \frac{\theta}{2} \right) = b$$

From the definition of $\cos$, we see that

$$\cos \left( \frac{\theta}{2} \right) = \frac{h}{r} \quad \text{or} \quad r \cos \left( \frac{\theta}{2} \right) = h$$

Hence, the area of the triangle is

$$\frac{1}{2} bh = \frac{1}{2} 2r \sin \left( \frac{\theta}{2} \right) r \cos \left( \frac{\theta}{2} \right)$$
\[ = \frac{1}{2} r^2 \cdot 2 \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) \]

Let's simplify this using the double angle formula.

\[ \sin(2x) = 2 \sin(x) \cos(x). \]

Letting \( x = \frac{\theta}{2} \), the above area formula can be rewritten as:

\[ \frac{1}{2} r^2 \sin(\theta). \]

Or writing it in terms of \( \sin \) and \( \cos \):

\[ \frac{1}{2} r^2 \sin \left( \frac{2\pi}{n} \right). \]

Summing the areas of \( n \) of these triangles, we get the following approximation to the area of the circle:

\[ A_n = \frac{1}{2} n r^2 \sin \left( \frac{2\pi}{n} \right). \]

The larger \( n \), the better the approximation \( A_n \) is to the area of the circle.
In fact, the area of the circle is \( \lim_{n \to \infty} A_n \). To compute this limit we need the fact that

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{(Prove it using L'Hopital's rule)}
\]

\[
\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{n}{2} \frac{r^2 \sin \left( \frac{2\pi}{n} \right)}{2} = \frac{r^2}{2} \lim_{n \to \infty} n \sin \left( \frac{2\pi}{n} \right)
\]

\[
= \frac{r^2}{2} \lim_{n \to \infty} 2\pi \sin \left( \frac{2\pi}{n} \right) \left( \frac{2\pi}{n} \right) = 2\pi r^2 \lim_{n \to \infty} \frac{\sin \left( \frac{2\pi}{n} \right)}{\frac{2\pi}{n}} \quad \text{note as } n \to \infty \quad \frac{2\pi}{n} \to 0
\]

\[
= \pi r^2 \cdot 1
\]

This shows the area of a circle is \( \pi r^2 \).