Problem 328, #16

Show that of all rectangles with a given area \( A \), the one with the smallest perimeter \( P \) is a square.

Consider the rectangle with sides of length \( l \) and width \( w \), respectively.

\[
\begin{align*}
A &= l \times w \\
P &= 2l + 2w \\
\end{align*}
\]

Must choose \( l \) and \( w \) such that \( P \) is maximized and \( A = l \times w \) holds.

Let’s write \( P \) as a function of one variable \( l \). Using \( A = l \times w \), we have \( w = \frac{A}{l} \).

So

\[
P(l) = 2l + 2 \frac{A}{l}
\]

To maximize \( P \) we find \( l \) such that \( P'(l) = 0 \)

\[
P'(l) = 2 - 2A \frac{1}{l^2}
\]

Since \( A = l \times w \), we have 

\[
0 = 2 - 2A \frac{1}{l^2}
\]

\[
l^2 = A
\]

\[
l = \sqrt{A}
\]

Since \( l \) and \( w \) are the same, the figure is a square.
Find points on the ellipse \(4x^2 + y^2 = 4\) that are farthest away from \((1,0)\).

Must maximize \((x-1)^2 + (y-0)^2\) where \((x,y)\) is on the ellipse.

Use ellipse equation to solve for \(y\) in terms of \(x\).

\[
4x^2 + y^2 = 4
\]

\[
y^2 = 4 - 4x^2
\]

\[
y = \sqrt{4 - 4x^2} = 2\sqrt{1-x^2}
\]

The distance \((x-1)^2 + y^2\) can now be written as a function of \(x\).

\[
d(x) = (x-1)^2 + 4 - 4x^2
\]

Or

\[
d(x) = -3x^2 - 2x + 5
\]

To find maximum, solve \(d'(x) = 0\).

\[
d'(x) = -6x - 2, \quad -6x - 2 = 0 \quad \text{so} \quad x = -\frac{1}{3}
\]

The corresponding \(y = 2\sqrt{1-x^2}\). So \(y = 2\sqrt{\frac{2}{3}} = \pm \frac{4\sqrt{2}}{3}\).

The points on the ellipse farthest from \((1,0)\) are \((-\frac{1}{3}, \pm \frac{4\sqrt{2}}{3})\).
Let's center the rectangle inscribed in the ellipse at the origin. If we label the right corner of the rectangle $(x, y)$, then the other corners must be labelled as indicated above. The lengths of the sides also are shown in the above figure. The area of the rectangle is

$$4xy$$

Since $x$ and $y$ are on the ellipse, they satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We have to choose the point $(x, y)$ on the ellipse that maximizes the area of the rectangle.
First let's write $y$ in terms of $x$ by using the equation of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = b \left( 1 - \frac{x^2}{a^2} \right)^{\frac{1}{2}}$$

Thus the area $A_{xy}$ can be written as a function of $x$,

$$A(x) = 4x \cdot b \left( 1 - \frac{x^2}{a^2} \right)^{\frac{1}{2}}$$

To find the $x$ value that maximizes the area, we take the derivative with respect to $x$, set it to zero, and solve.

$$A'(x) = 4b \left[ \frac{x}{2} \left( 1 - \frac{x^2}{a^2} \right)^{-\frac{1}{2}} \cdot \left( -\frac{2x}{a^2} \right) + \left( 1 - \frac{x^2}{a^2} \right)^{\frac{1}{2}} \right]$$

To find $x$ such that $A(x) \to 0$, we need to do some algebra.
Let's rewrite $A'(x)$ as

$$A'(x) = 4b \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \left[-\frac{x^2}{a^2} + 1 - \frac{x^2}{a^2}\right]$$

or

$$A'(x) = 4b \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \left[-2x^2 + 1\right]$$

Now the only way $A'(x)$ could be 0 is if

$$1 - \frac{2x^2}{a^2} = 0 \quad \text{or} \quad x^2 = \frac{a^2}{2} \quad \text{or} \quad x = \frac{a}{\sqrt{2}}$$

We have the optimal $x$. Let's use the ellipse equation to get the corresponding $y$.

$$y = b \left(1 - \frac{(\frac{a}{\sqrt{2}})^2}{a^2}\right) \frac{1}{2}$$

$$y = b \left(1 - \frac{1}{2}\right)^{\frac{1}{2}}$$

$$y = \frac{b}{\sqrt{2}}$$

The maximal area is thus

$$4xy = 4 \frac{a}{\sqrt{2}} \frac{b}{\sqrt{2}} = 2ab$$
P 329, #29

A right circular cylinder is inscribed in a sphere of radius \( r \). Find the largest volume for the cylinder.

Formulas we need:
- Volume of cylinder: \( V = \pi r^2 h \), \( r \) radius of cylinder
- Equation of sphere: \( x^2 + y^2 + z^2 = r^2 \)

The cylinder is determined by considering the rectangle formed in the \( x \) and \( z \) plane.

Key:
- \( P(x,0,z) \) point on sphere and cylinder.
- \( h = 2z \)
- \( h = \rho = x \)

\[
V = \pi r^2 h
\]
\[
V = \pi x^2 z
\]
\[
V = 2\pi x^2 z
\]

\[
V(x) = 2\pi x^2 (r^2 - x^2)^{1/2}
\]

\[
V(x) = 2\pi \left[ \frac{1}{2} x^2 (r^2 - x^2)^{1/2} + 2x (r^2 - x^2)^{1/2} \right]
\]

\[
= 2\pi \left[ (r^2 - x^2)^{1/2} \left( x^3 + 2x (r^2 - x^2) \right) \right]
\]
\[ V'(x) = 0 \Rightarrow 2\pi r^2 - 3x^3 = 0 \]

\[ 2\pi r^2 = 3x^3 \]

\[ \frac{2}{3} r^2 = x^2 \]

Optimal \( x \) in terms of radius \( r \)

\[ \rightarrow \frac{\sqrt{2}}{\sqrt{3}} r = x \]

Find \( z \) in terms of \( r \)

\[ z = (r^2 - x^2)^{\frac{1}{2}} \]

\[ z = \left( r^2 - \frac{2}{3} r^2 \right)^{\frac{1}{2}} \]

\[ z = r \left( 1 - \frac{1}{3} \right)^{\frac{1}{2}} \]

\[ z = r \left( \frac{\sqrt{2}}{\sqrt{3}} \right) \]

\[ z = r \frac{\sqrt{2}}{\sqrt{3}} \]

Maximum volume of the inscribed cylinder

\[ V = \frac{4\pi r^3}{3\sqrt{3}} \]